

INTEGRAL OF EXPONENT OF A POLYNOMIAL IS A GENERALIZED HYPERGEOMETRIC FUNCTION OF THE COEFFICIENTS OF THE POLYNOMIAL

A. V. STOYANOVSKY

ABSTRACT. We show that the integral $\int e^{S(x_1, \dots, x_n)} dx_1 \dots dx_n$ for an arbitrary polynomial S , satisfies a generalized hypergeometric system of differential equations in the sense of I. M. Gelfand et al.

Integrating over a linear space with exponential weight is an important procedure in statistical and quantum physics. However, till recent time, the integral of exponent of a polynomial of several variables has been known only for quadratic polynomials (Gaussian integral).

The pioneering works on integrating exponents of non-quadratic polynomials are due to V. V. Dolotin, A. Yu. Morozov, and Sh. R. Shakirov [1–5]. In these works one considers homogeneous polynomials (forms). In the paper [5], in order to compute integral of exponent of a form, one uses differential equations satisfied by the integral as a function of the coefficients of the form, and one also uses invariant theory. It is shown that in several particular cases, the integral is a generalized hypergeometric function in the sense of I. M. Gelfand *et al.* [6] of algebraic invariants of a form. The authors of [1–5] consider the integral as an invariant of the special linear group, and use the term “integral discriminant” for it.

The purpose of the present note is to make the following simple observation: the system of differential equations satisfied by the integral of exponent of an arbitrary (not necessarily homogeneous) polynomial, as a function of the coefficients of the polynomial, coincides with the generalized hypergeometric system (GHS) [6], so that the integral is a generalized hypergeometric function of the coefficients of the polynomial.

Indeed, consider the integral

$$(1) \quad Z = \int e^{S(x)} dx,$$

Partially supported by the grant RFBR 10-01-00536.

where $x = (x_1, \dots, x_n)$, $dx = dx_1 \dots dx_n$,

$$(2) \quad S(x) = \sum_{k \in K} c_k x^k,$$

$k = (k_1, \dots, k_n)$, $x^k = x_1^{k_1} \dots x_n^{k_n}$, $S(x)$ is a polynomial with complex coefficients, and integration goes over an n -dimensional real contour in \mathbb{C}^n , on which the function $e^{S(x)}$ rapidly decreases at infinity. Assume that the degrees of the monomials $k \in K$ span the space \mathbb{C}^n . One has a surjective linear map $\pi : \mathbb{C}^K \rightarrow \mathbb{C}^n$, $\sum_{k \in K} n_k e_k \mapsto \sum_{k \in K} n_k \cdot k$, where e_k is the basis vector in \mathbb{C}^K corresponding to k . Consider the lattice $B = \mathbb{Z}^K \cap \text{Ker } \pi \subset \mathbb{C}^K$, the subspace $L = \text{Ker } \pi$ spanned by B , and the vector $\alpha = (-1, -1, \dots, -1) \in \mathbb{C}^n \simeq \mathbb{C}^K / L$. Let $A \subset (\mathbb{C}^K)'$ be the annihilator of L , $A \simeq (\mathbb{C}^n)'$.

Theorem. *The integral (1) satisfies the generalized hypergeometric system of equations on the space \mathbb{C}^K corresponding to the lattice B and to the parameter α .*

Proof. Recall that GHS reads ([6], formulas (0.5),(0.6))

$$(3) \quad \prod_{k: n_k > 0} \left(\frac{\partial}{\partial c_k} \right)^{n_k} Z = \prod_{k: n_k < 0} \left(\frac{\partial}{\partial c_k} \right)^{-n_k} Z \text{ for all } (n_k) \in B,$$

$$(4) \quad \sum_{k \in K} a_k c_k \frac{\partial Z}{\partial c_k} = \langle a, \alpha \rangle Z \text{ for all } a \in A.$$

Equality (3) is obvious. To check (4), it suffices to put $a_k = k_i$ for a fixed i , $1 \leq i \leq n$. Then the left hand side of (4) equals

$$\int \sum_k k_i c_k x^k e^{S(x)} dx = \int x_i \frac{\partial}{\partial x_i} e^{S(x)} dx = - \int e^{S(x)} dx$$

(integration by parts), Q. E. D.

The author is grateful to V. V. Dolotin for numerous discussions, and to Sh. R. Shakirov for his talk about the paper [5] at the author's seminar at the Independent Moscow University.

Remark. It turned out that in the paper [7], I. M. Gelfand and M. I. Graev pointed out that integral of exponent of a sum of monomials with arbitrary complex powers of variables formally satisfies generalized hypergeometric system of equations, so our result is not new in this respect. An essential difference with our setup is that the authors of [7] consider integration contours in $(\mathbb{C} \setminus 0)^n$ which do not go to infinity. In the case of exponent of a usual polynomial, integral over such contour vanishes.

REFERENCES

- [1] V. Dolotin, On discriminants of polylinear forms, *Izvestiya Mathematics*, vol. 62, No 2, 3–34 (1998); arXiv:alg-geom/9511010.
- [2] V. Dolotin, On invariant theory, arXiv:alg-geom/9512011.
- [3] V. Dolotin, QFT's with action of degree 3 and higher and degeneracy of tensors, arXiv:hep-th/9706001.
- [4] V. Dolotin and A. Morozov, Introduction to non-linear algebra, World Scientific, 2007, arXiv:hep-th/0609022.
- [5] A. Morozov and Sh. Shakirov, Introduction to integral discriminants, arXiv:0903.2595 [math-ph].
- [6] I. M. Gelfand, M. I. Graev, V. S. Retakh, General hypergeometric systems of equations and series of hypergeometric type, *Uspekhi Mat. Nauk* (Russian Math. Surveys), vol. 47, No. 4, 3–82 (1992).
- [7] I. M. Gelfand and M. I. Graev, GG-functions and their relations to A-hypergeometric functions, arXiv:math/9905134 [math.AG].

RUSSIAN STATE UNIVERSITY OF HUMANITIES

E-mail address: alexander.stoyanovsky@gmail.com